We may again associate with each substitution of I the index (modulo $p^\alpha q^\beta$) of the power into which the substitution transforms all the substitutions of C. If this is done Legendre's law of quadratic reciprocity says that the substitution which corresponds to $k \equiv p+q$ modulo $p^\alpha q^\beta$ is positive unless p and q are both of the form $4n+3$. It is clear that the positive substitutions of I correspond to numbers which are either quadratic residues of both p and q, or quadratic non-residues of both p and q. The negative substitutions of I correspond to numbers which are quadratic residues of only one of the two numbers p, q.

In abstract group theory the following direct application of the law of quadratic reciprocity is evident from the preceding developments. If the group of isomorphisms I of the cyclic group of order p^α is represented as a number group modulo p^α all the numbers which are in the subgroup of order $\frac{p^{\alpha-1}(p-1)}{2}$ are quadratic residues of p while the remaining numbers are non-residues. All the numbers of I may be represented by primes in an infinite number of ways since they are only determined with respect to modulus p^α. Legendre's law of quadratic reciprocity states that if q is any prime in the subgroup of I whose order is $\frac{p^{\alpha-1}(p-1)}{2}$ then will the numbers which are congruent to p mod q be in the subgroup of order $\frac{q^{\beta-1}(q-1)}{2}$ of the group of isomorphisms of the cyclic group of order q^β, unless both p and q are of the form $4n+3$.

Since the subgroup of order p^{α_1}, $\alpha_1 < \alpha$, contained in I is composed of the numbers which are congruent to 1 modulo $p^{\alpha-\alpha_1}$, it follows that the subgroup of order $\frac{p^{\alpha_1}(p-1)}{d}$ contained in I is composed of the numbers which are congruent modulo $p^{\alpha-\alpha_1}$ to the numbers of the subgroup of order $\frac{p-1}{d}$. When $d=2$ the latter numbers are the quadratic residues of p modulo p. If we multiply these numbers by the numbers in the subgroup of order p^{α_1} the products are still quadratic residues of p. By making $\alpha_1 = \alpha - 1$ we have another proof of the theorem that a quadratic residue of p if also a quadratic residue of p^α. Another particular case is that *every subgroup of I whose order is divisible by $p^{\alpha-1}$ includes the numbers which are congruent to any of its numbers modulo p.*

While we know some of the properties of the numbers in the subgroups of I, little is known in reference to what particular numbers occur in these subgroups. It is evident that $p^\alpha - 1$ occurs in every subgroup of even order. That is, when the group constituted by the quadratic residues of p^α is of even order, -1 is a quadratic residue of p and vice versa. This follows directly from the fact that -1 corresponds to the operator of order 2 in I. The numbers 1 and -1 are the only ones whose orders are independent of the value of the modulus.

As has been observed above the numbers whose orders are given powers of p can also be directly written down. The determination of the numbers whose orders divide $p-1$ presents difficulties which have not yet been surmounted. If we knew the latter numbers and their orders, we would know the orders of all the numbers, since the order of the product of two such numbers is the product of their orders. In the case when $p=3$, the operators whose orders divide $p-1$ are 1 and p^a-1. Hence, in this case, the order of every number is known. In particular, the primitive roots of p^a are the products of p^a-1 into the numbers of the form $1+kp$ (k being prime to p), as is well known.

Even the number 2 presents difficulties which have not been overcome. That is, we do not know the order of the operator which transforms each operator of C into its square. The well known number theory results along this line; that is, those which relate to the quadratic character of 2, may be stated in group theory language as follows: In the number group formed by the natural numbers which are prime to p and less than p^a, the order of 2 divides $\dfrac{p^{a-1}(p-1)}{2}$ only when p is of the form $8n\pm1$. When p is not of this form the order of 2 must involve the highest power of 2 contained in $p-1$.

A SHORT PROOF FOR THE NUMBER OF TERMS IN A DETERMINANT WHICH ARE INDEPENDENT OF THE ELEMENTS OF THE PRINCIPAL DIAGONAL.

By ORLANDO S. STETSON, Syracuse University.

The problem of finding the number of terms, $\varphi(n)$, in the given determinant which are independent of the elements of the principal diagonal may be reduced to the question of finding the number of terms in the expansion of the *invertebrate* determinant Δ_n (second formula for $k=n$, MONTHLY, 1904, page 167). Hence

$$\varphi(n)=n!-n(n-1)!+\frac{n(n-1)}{1.2}(n-2)!-\frac{n(n-1)(n-2)}{1.2.3}(n-3)!$$

$$+\dots\dots+(-1)^{n-1}\frac{n(n-1)(n-2)\dots\dots3.2}{1.2.3\dots\dots(n-1)}(1)!+(-1)^n\frac{n(n-1)(n-2)\dots\dots3.2.1}{1.2.3\dots\dots n}.$$

Removing from each of the terms the factor $n!$ and noticing that the first two terms are equal but opposite in sign, we have

$$\varphi(n)=n!\left[\frac{1}{1.2}-\frac{1}{1.2.3}+\frac{1}{1.2.3.4}-\dots\dots+(-1)^n\frac{1}{1.2.3\dots\dots n}\right].$$

NOTE ON THE nth DERIVATIVE OF A DETERMINANT WHOSE CONSTITUENTS ARE FUNCTIONS OF A GIVEN VARIABLE.*

By W. J. RUSK. Grinnell, Iowa.

Let the determinant be $D=(a_1 b_2 \ldots\ldots l_r)$ where the a, b, ..., l are functions of a variable t. Then

$$\frac{dD}{dt}=(a_1' b_2 \ldots\ldots l_r)+\ldots\ldots+(a_1 b_2 \ldots\ldots l_r').$$

$$\frac{d^2D}{dt^2}=(a_1'' b_2 \ldots\ldots l_r)+\ldots\ldots+(a_1 b_2 \ldots\ldots l_r'')$$

$$+2(a_1' b_2' c_3 \ldots\ldots l_r)+\ldots\ldots+2(a_1 b_2 \ldots\ldots k'_{r-1} l_r').$$

Consider now the expansion

$$(a+b+c+\ldots\ldots+l)^2=a^2 b^0 c^0 \ldots\ldots l^0+a^0 b^2 c^0 \ldots\ldots l^0+2a^1 b^1 c^0 \ldots\ldots l^0+\ldots\ldots$$

If we interpret the power of a^n as the nth derivative of a and write instead of $a^2 b^0 c^0 \ldots\ldots l^0$ the expression $(a_1'' b_2 c_3 \ldots\ldots l_r)$, etc., we have, symbolically,

$$\frac{d^2D}{dt^2}=(a+b+c+\ldots\ldots+l)^2.$$

Suppose now that, symbolically,

$$\frac{d^{n-1}D}{dt^{n-1}}=(a+b+c+\ldots\ldots+l)^{n-1}=\sum\frac{(n-1)!}{\alpha!\,\beta!\ldots\ldots\lambda!}a^\alpha b^\beta \ldots\ldots l^\lambda,$$

where $\alpha+\beta+\gamma+\ldots\ldots+\lambda=n-1$.

Now suppose the symbolic form be interpreted as before and another differentiation with respect to t carried out. Then the coefficient of $a^{\alpha'} b^{\beta'} \ldots\ldots l^{\lambda'}$ will be

$$C=\frac{n!}{\alpha'!\,\beta'!\ldots\ldots\lambda'!},$$

where $\alpha'+\beta'+\ldots\ldots+\lambda'=n$. For this term can be obtained by differentiation from terms with exponents one less than α', β', ..., or λ', and its coefficient will be

$$\frac{(n-1)!}{(\alpha'-1)!\,\beta'!\ldots\ldots\lambda'!}+\frac{(n-1)!}{\alpha'!\,(\beta'-1)!\ldots\ldots\lambda'!}+\ldots\ldots=C.$$

*Presented to the American Mathematical Society (Chicago), April, 1904.

ON THE CYCLOTOMIC FUNCTION.*

By DR. L. E. DICKSON, The University of Chicago.

1. If $p^n=1$ and $p^m\neq 1$ $(m<n)$, p is called a primitive nth root of unity. Let $Q_n(x)$ be the equation whose roots are the various nth primitive roots of unity without repetition. Let $n=\nu p^r$, ν not being divisible by the prime p. We first prove that

$$(1)\qquad Q_n(x)=Q_\nu(x^{p^r})\div Q_\nu(x^{p^{r-1}}).$$

To show that the division is exact, let $\xi_1, \ldots, \xi_e$ be the distinct νth roots of unity. Then $\xi_1{}^p, \ldots, \xi_e{}^p$ differ only as to order from $\xi_1, \ldots, \xi_e$. Hence

$$Q_\nu(x^{p^r})=\prod_{i=1}^{e}(x^{p^r}-\xi_i{}^p),\quad Q_\nu(x^{p^{r-1}})=\prod_{i=1}^{e}(x^{p^{r-1}}-\xi_i).$$

Since $y-\xi$ divides $y^p-\xi^p$, the division (1) is exact. If the value x makes the quotient vanish, then $(x^{p^r})^m=\xi_i{}^m=1$ if and only if m is a multiple of ν, while $x^{\nu p^{r-1}}\neq 1$; hence x is a primitive nth root of unity.

We employ (1) as a recursion formula to determine $Q_n(x)$. As a permanent notation, set $n=p_1{}^{r_1}p_2{}^{r_2}\ldots p_s{}^{r_s}$, where $p_1, \ldots, p_s$ are distinct primes. Then

$$(2)\quad Q_n(x)=Q_\mu(x^{p_1^{r_1}p_2^{r_2}})Q_\mu(x^{p_1^{r_1-1}p_2^{r_2-1}})\div Q_\mu(x^{p_1^{r_1}p_2^{r_2-1}})Q_\mu(x^{p_1^{r_1-1}p_2^{r_2}}),$$

where $\mu=n\div p_1{}^{r_1}p_2{}^{r_2}$. In general, for $N=p_1{}^{r_1}p_2{}^{r_2}\ldots p_\sigma{}^{r_\sigma}$, we get

$$(3)\qquad Q_n(x)=\frac{Q_{n/N}(x^N)\,\Pi Q_{n/N}(x^{N/p_ip_j})\ldots}{\Pi Q_{n/N}(x^{N/p_i})\,\Pi Q_{n/N}(x^{N/p_ip_jp_k})\ldots},$$

in which $i, j, \ldots$ range from $1, \ldots, \sigma$. Now $Q_1(x)=x-1$. For $\sigma=s$, $N=n$, and (3) becomes

$$(4)\qquad Q_n(x)=\frac{(x^n-1)\,\Pi(x^{n/p_ip_j}-1)\ldots}{\Pi(x^{n/p_i}-1)\,\Pi(x^{n/p_ip_jp_k}-1)\ldots},$$

where in the denominator the products extend over the combinations 1, 3, 5, ... at a time of $p_1, \ldots, p_s$; in the numerator, 2, 4, ... at a time.

Conversely, (1) follows from (4). The terms of (4) in which p_1 does not enter explicitly combine into $Q_\nu(x^{p_1^{r_1}})$; those in which p_1 enters explicitly combine into $1\div Q_\nu(x^{p_1^{r_1-1}})$.

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*Read before the American Mathematical Society, Chicago, April 22, 1905.

2. The usual proof* of (4) is essentjally only a verification. Since $Q_d(x)=0$ gives all the primitive dth roots of unity withour repetition, we have

$$(5) \qquad x^n-1=\Pi Q_d(x), \quad x^{n/p_1}-1=\Pi Q_\delta(x), \ldots$$

where d ranges over all the divisors of n; δ over those of n/p_1. When the products (5) are substituted in the second member of (4), every Q cancels except Q_n. In fact, if $d=p_1^{a_1}\ldots p_t^{a_t}p_{t+1}^{r_{t+1}}\ldots p_s^{r_s}$, where $t>0$ and each $a_i<r_i$, Q_d divides exactly $A=1+{}_tC_2+{}_tC_4+\ldots$ terms of the numerator of (4) and exactly $B={}_tC_1+{}_tC_3+\ldots$ terms of the denominator, ${}_tC_k$ being the number of combinations of t things k at a time. But $A-B=(1-1)^t=0$.

3. From equation (4) follows as a corollary the important formula

$$(6) \qquad \phi(n)=n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\ldots\left(1-\frac{1}{p_s}\right),$$

where $\phi(n)$ denotes the number of positive integers not greater than n and relatively prime to n. Indeed, if ρ is a primitive nth root of unity, ρ^m is likewise if and only if m is relatively prime to n. But the degree of (4) evidently equals the right member of (6).

It follows from (4) that the polynomial $Q_n(x)$ has integral coefficients.

There are various proofs of the theorem that $Q_n(x)$ is algebraically irreducible, *i. e.*, can not be expressed as a product of polynomials in x with rational coefficients.†

4. Theorem. *For an integer x, the greatest common divisor g of $Q_n(x)$ and $x^{n/p_1}-1$ is 1 or p_1. If $g=p_1$, Q_n is not divisible by p_1^2 unless $n=p_1=2$, $x\equiv 3$ (mod4), whence $Q_n=x+1$.*

Dividing the first equation (5) by the second, we get

$$(7) \qquad (x^{n/p_1})^{p_1-1}+(x^{n/p_1})^{p_1-2}+\ldots+x^{n/p_1}+1=Q_n(x).P(x),$$

$P(x)$ being a polynomial in x with integral coefficients. When the left member of (7) is divided by $x^{n/p_1}-1$, the remainder is 1 or p_1. Hence $g=1$ or p_1.

Let $g=p_1$, so that $x^{n/p_1}-1=kp_1$, k an integer. Substituting kp_1+1 for x^{n/p_1} in (7), we obtain $p_1+\frac{1}{2}p_1(p_1-1)kp_1+$terms in p_1^2. This is not divisible by p_1^2 if $p_1>2$, nor if $p_1=2$ and k is even. If $p_1=2$ and $k=2l+1$, then $x^{n/2}=4l+3$, whence $n/2$ must be odd and $x\equiv 3$ (mod4). Suppose that $n>2$ and $n/2=p^r p_3^{r_3}\ldots p_s^{r_s}=m=$odd. Performing in (4) the divisions of the type $(x^{2a}-1)\div(x^a-1)$, we get

$$(8) \qquad \frac{x^m+1}{x^{m/p}+1}\cdot\prod_{i=3}^{s}\frac{x^{m/pp_i}+1}{x^{m/p_i}+1}\cdot\prod_{i,j=3}^{s}\frac{x^{m/p_ip_j}+1}{x^{m/pp_ip_j}+1}\cdot\prod_{i,j,k}\frac{x^{m/pp_ip_jp_k}+1}{x^{m/p_ip_jp_k}+1}\ldots$$

*On the general principle of the inversion involved, see Dedekind, *Crelle*, Vol. 54 (1857), pp. 1-26; Dirichlet-Dedekind, *Zahlentheorie*, p. 362; Bachmann, *Kreistheilung*, 1872, pp. 8-11, 16, and *Zahlentheorie*, I, pp. 40-42.

†See Bachmann, *Kreistheilung* (Leipzig. Teubner, 1872), pp. 31-43.

Since the exponents are all odd, each fraction or its inverse equals $1+f$, f containing an even number of powers of x. Hence Q is odd (cf. §5).

5. Theorem. *For $n=p_1^{r_1}.....p_s^{r_s}$ and x an integer, $Q_n(x)$ is divisible by p_1 if and only if x belongs to the exponent $\nu=n/p_1^{r_1}$ modulo p_1; in the contrary case, $Q_n(x)\equiv 1 \pmod{p_1}$.*

By Fermat's theorem, $x^{p_1}\equiv x \pmod{p_1}$. Hence by (1), $Q_n(x)\equiv 1 \pmod{p_1}$ unless $Q_\nu(x)\equiv 0$. Now $Q_\nu(x)$ divides algebraically the function

$$(x^\nu-1)\div(x^{\nu/p_i}-1)=(x^{\nu/p_i})^{p_i-1}+.....+x^{\nu/p_i}+1 \qquad (1<i\leqq s).$$

Hence if $x^{\nu/p_i}\equiv 1 \pmod{p_1}$, there is an integer k such that $kQ_\nu(x)\equiv p_i \pmod{p_1}$; whence Q_ν is not congruent to 0 (mod p_1). There remains the case in which x^{ν/p_i} is not congruent to 1 (mod p_1) for $i=2$, ..., s. If $x^\nu\equiv 1 \pmod{p_1}$, x belongs to the exponent ν modulo p_1 and $Q_\nu\equiv 0$; if $x^\nu-1$ is not congruent to 0, its divisor Q_ν is not congruent to 0 (mod p_1).

Example. For $n=2.3.7$, formula (8) gives

$$Q_{42}(x)=\frac{(x^{21}+1)\,(x+1)}{(x^7+1)\,(x^3+1)}=x^{12}+x^{11}-x^9-x^8+x^6-x^4-x^3+x+1.$$

Thus $Q_{42}\equiv 1$ (mod 2 or 3); $Q_{42}(x)\equiv 1 \pmod 7$ if $x\equiv 0$, -1 or $x^3\equiv +1$; but $Q_{42}(x)\equiv 0 \pmod 7$ if $x^2-x+1\equiv 0 \pmod 7$, *i. e.*, if x belongs to the exponent $\nu=6$.

Corollary. No one of the prime factors of n except the greatest can divide $Q_n(x)$.

6. Theorem. *If x is a positive integer >1, $Q_n(x)$ has a prime factor not dividing x^m-1 $(m<n)$, except in the cases $n=2$, $x=2^k-1$ $(k\geqq 2)$; and $n=6$, $x=2$.*

If $n=p^r$, $Q=y^{p-1}+.....+y+1>p$, where $y=x^{p^{r-1}}$, so that the theorem follows from §4. We suppose henceforth that $n=p_1^{r_1}.....p_s^{r_s}$, $s\geqq 2$.

In view of §1, $Q_n=A/B$, where

$$A=\frac{x^n-1}{x^{n/p_1}-1}\cdot\Pi\frac{x^{n/p_ip_j}-1}{x^{n/p_1p_ip_j}-1}.....,\ B=\Pi\frac{x^{n/p_i}-1}{x^{n/p_1p_i}-1}\cdot\Pi\frac{x^{n/p_ip_jp_k}-1}{x^{n/p_1p_ip_jp_k}-1}.....,$$

in which i, j, k, ... run from 2 to s. Now $x^{a(k-1)}<(x^{ka}-1)\div(x^a-1)<2x^{a(k-1)}$. Hence $Q_n>\alpha/\beta$, where α is the result of retaining only the first term of each division in A, β the result of taking twice the first term of each division in B. The number of factors 2 introduced in B is ${}_{s-1}C_1+{}_{s-1}C_3+.....=2^{s-2}$. The exponent of x in α/β is evidently the degree $\phi(n)$ of Q_n. Let $x^{n/p_1p_2\cdots p_s}=y$. Hence

$$Q_n(x)>y^{(p_1-1)\cdots(p_s-1)}\div 2^{2^{s-2}},$$

y an integer>1. In view of §§4-5, it suffices to prove that $Q_n>p_1$, the greatest of the primes p_i, the case $n=6$, $x=2$ being an exception. For $s>2$, we have $p_1\geqq 5$, $y^{p_1-1}>2p_1$, the latter being true for $y=2$. Hence

$$Q_n > (2p_1)^{(p_2-1)\dots(p_s-1)} \div 2^{2^{s-2}} > p_1,$$

since at least $s-2$ of the primes $p_2, \dots, p_s$ exceed 2, so that the exponent is $\geqq 2^{s-2}$. For $s=2$, we have $p_1 \geqq 3$, $Q_n > \frac{1}{2} y^{(p_1-1)(p_2-1)} \geqq p_1$ unless $p_1=3$, $y^{p_2-1}=2$, whence $p_2=2$, $y=2$, $n=6$, $x=2$.

Corollary. *If x is a positive integer >1, x^n-1 has a prime factor not dividing x^m-1 $(m<n)$, except in the cases $n=2$, $x=2^k-1$; $n=6$, $x=2$.*

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

Problems 219, 220 were also solved by L. E. Newcomb. No. 222 was also solved by A. H. Holmes.

223. Proposed by THEODORE L. DE LAND, Office of the Secretary of the Treasury, Washington, D.C.

An officer in the Treasury Department assigned three clerks to count a lot of silver dollars and when finished noted that there was an apparent difference in their efficiency; and, to determine the fact, gave to each a similar lot of the same amount to count, the only record made at the time being that A to count his lot alone, took three weeks longer, B took two weeks longer, and C took one week longer than it took for all working together to count the first lot. The best counter, on the record made, was given an efficiency mark of 93 on the scale of 100. What efficiency mark should, on the record, be given to each of the other two counters?

Solution by the PROPOSER.

Let x=the time for A, B, and C working together to finish one lot.

Then $x+3$=the time for A to finish one lot working alone;

$x+2$=the time for B to finish one lot working alone; and

$x+1$=the time for C to finish one lot working alone.

Then $\dfrac{1}{x}$=what A, B, and C can do in one week working together;

$\dfrac{1}{x+3}$=what A can do in one week working alone;

$\dfrac{1}{x+2}$=what B can do in one week working alone; and

$\dfrac{1}{x+1}$=what C can do in one week working alone.

Equating like terms we have,

$$\frac{1}{x} = \frac{1}{x+3} + \frac{1}{x+2} + \frac{1}{x+1} \quad \dots\dots (1).$$

Reducing, we have,

$$x^3+3x^2-3=0 \text{ .... } (2).$$

By Horner's method, we have from equation (2), $x=0.879385+$. Therefore

$$\frac{1}{x+3}=\frac{1}{3.839385};\quad \frac{1}{x+2}=\frac{1}{2.879385};\ \text{and}\ \frac{1}{x+1}=\frac{1}{1.879385}.$$

It is evident that C is the best clerk and was given the 93% on the efficiency record. The records should be inversely proportional to the time expended for equivalent work. In order to compare C and B, and C and A, we have

$$x+2 : x+1=93\% : B\text{'s mark};$$
$$x+3 : x+1=93\% : A\text{'s mark};$$

and therefore,

$$2.879385 : 1.879385=93\% : 60.70\%=B\text{'s mark; and}$$
$$3.879385 : 1.879385=93\% : 45.05\%=A\text{'s mark}.$$

Thus, if C were given on the efficiency record 93%, A should be given 45.05%, and B should be given 60.70%.

Also solved by G. B. M. Zerr, S. A. Corey, G. W. Greenwood, F. D. Whitlock, R. D. Carmichael, A. H. Holmes, and J. Scheffer.

224. Proposed by G. W. GREENWOOD, M. A. (Oxon), Lebanon, Ill.

Show that, if none of the quantities x, y, z is zero, the result of eliminating them from

$$(x+y)(x+z)=bcyz \text{ .... } (1),$$
$$(y+z)(y+x)=cazx \text{ .... } (2),$$
$$(z+x)(z+y)=abxy \text{ .... } (3),$$

is $\begin{vmatrix} \pm a, & 1, & 1 \\ 1, & \pm b, & 1 \\ 1, & 1, & \pm c \end{vmatrix}=0.$ [Oxford, 1896.]

Solution by C. H. MILLER, West Point, N. Y., and the PROPOSER.

By multiplying the second equation by the third, dividing by the first, and transposing, we obtain

$$\pm ax+g+z=0.$$

From this, and two similar equations, we get the required elimininant.

Also solved by J. B. Faught, G. B. M. Zerr, R. D. Carmichael, J. Scheffer, and J. O. Mahoney.

225. Proposed by H. M. ARMSTRONG, Cooch's Bridge, Delaware.

If $\alpha=ax+cy+bz$ .... (1), $\beta=cx+by+az$ .... (2), $\gamma=bx+ay+cz$ .... (3), show that $\alpha^3+\beta^3+\gamma^3-3\alpha\beta\gamma=(a^3+b^3+c^3-3abc)(x^3+y^3+z^3-3xyz)$.

Solution by the PROPOSER.

The required result follows directly from the equality,

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \gamma & \alpha & \beta \\ \beta & \gamma & \alpha \end{vmatrix} = \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} \cdot \begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix}.$$

Also solved by J. B. Faught, G. B. M. Zerr, G. W. Greenwood, Grace M. Bareis, J. O. Mahoney, F. D. Posey, F. O. Whitlock, J. Scheffer.

*** Dr. L. E. Dickson points out that a similar theorem holds for any determinant whose matrix is the body of a multiplication-table of a finite group.

GEOMETRY.

251. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

Represent the vertices of any regular polygon by the consecutive numbers $1, 2 \ldots p \ldots q \ldots r \ldots n$. To find the sides and area of the triangle formed by joining p, q, and r.

Solution by G. W. GREENWOOD, M. A. (Oxon), Lebanon, Ill., and A. H. HOLMES, Brunswick, Me.

The central angles subtended by the chords (pq) and (qr) are respectively,

$$2(q-p)\frac{\pi}{n} \text{ and } 2(r-q)\frac{\pi}{n}.$$

The angle pqr is found to be $\pi-(r-p)\frac{\pi}{n}$. Hence the required area is

$$\tfrac{1}{2}.pq.qr.\sin\angle pqr = 2a^2\sin(q-p)\frac{\pi}{n}.\sin(r-q)\frac{\pi}{n}.\sin(r-p)\frac{\pi}{n},$$

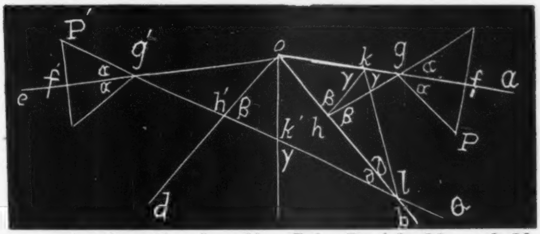
where a is the radius of the circum-circle of the polygon.

252. Proposed by FREDERICK R. HONEY, Ph. B., Trinity College, Hartford, Conn.

Two plane mirrors form an angle which is less than 45°. Any two points are assumed within this angle in a plane perpendicular to the intersection of the mirrors. A ray of light passes through one point, and after being reflected twice at each mirror, it passes through the second point. Find the path of the ray.

Solution by R. A. WELLS, Westminster College, Fulton, Mo.; THEODORE LINQUIST, Wahpeton, N. D.; and the PROPOSER.

Let oa and ob represent the mirrors; and P and Q the assumed points. Draw oc, od, and oe, making each of the angles boc, cod, and doe equal to aob. Draw Pf perpendicular to oa. Make $of'=of$; and draw $f'P'$ perpendicular to oe and equal to Pf. Draw QP', intersecting ob at l, oc at k', od at h', and oe at g'. Make $og=og'$; $oh=oh'$; $ok=ok'$. Join Pg, gh, hk, and kl. $PghklQ$ is the path of the ray.



The Greek letters indicate the equality of certain angles, and will assist the reader in the demonstration.

Also solved by G. W. Greenwood.

The following contributors sent in solutions to this department too late for credit in the last issue: G. B. M. Zerr solved 245; Theodore Linquist, 248 and 249; A. H. Holmes, 248, 249, and 250.

253. Proposed by SAM I. JONES, Gunter Bible College, Gunter, Texas.

The number of cubic inches contained by two equal opposite spherical segments, together with the number of cubic inches contained by the cylinder included between these segments, is 600; if this be $\frac{2}{3}$ of the number of cubic inches contained by the whole sphere, find the height of the cylinder.

Solution by THEODORE LINQUIST, Wahpeton, N. Dak.; G. W. GREENWOOD, M. A. (Oxon), Lebanon, Ill., and A. H. HOLMES, Brunswick, Me.

Let R=the radius of the sphere, and $2h$ the altitude of the cylander. Then $R-h$=the altitude of the segment of the sphere, and $\sqrt{(R^2-h^2)}$ is the radius of the base of the segment and the radius of the cylinder.

The volume of the two segments$=2[\frac{1}{6}\pi(R-h)^3+\frac{1}{2}\pi(R-h)(R^2-h^2)]$, and the volume of cylinder$=2\pi h(R^2-h^2)$.

$\therefore \frac{4}{3}\pi(R^3-h^3)$=the volume of the segments, and the cylinder$=\frac{2}{3}(\frac{4}{3}\pi R^3)$, by the conditions of the problem.

$\therefore 3h^3=R^3$. $\therefore \frac{4}{3}\pi(R^3-h^3)=\frac{8}{3}\pi h^3=600$, by the conditions of the problem.

$\therefore 2h=2\sqrt[3]{(225/\pi)}$.

Also solved by J. Scheffer.

CALCULUS.

191. Proposed by J. E. SANDERS, Hackney, Ohio.

A fly goes along a radius of a moving carriage wheel from center to circumference while the wheel makes n revolutions. If each move uniformly, what is the equation to the curve described by the fly in space, and what is its length when the wheel has made $1/m$ of a revolution?

Solution by G. W. GREENWOOD, M. A. (Oxon), Professor of Mathematics, McKendree College, Lebanon, Ill.

Take the path of the center of the wheel as x-axis, and the initial point as origin. Let the fly move on a radius making, initially, an angle ϕ with this axis. Denote the radius by a. Let C be the position of the center of the wheel, and P be that of the fly after the wheel has turned through an angle ω. Then

$$OC=a\omega,\ CP=\frac{a\omega}{2n\pi},$$

and the coördinates of the position of P are

$$x=a\omega\left(1+\frac{\cos(\omega+\phi)}{2n\pi}\right),\ y=\frac{a\omega\sin(\omega+\phi)}{2n\pi}.$$

*** An excellent solution was received from Professor Zerr. He takes the horizontal line on which the wheel travels as the x-axis, and gets for the equation of the path of the fly,

$$x=a\theta\left(1-\frac{\sin\theta}{2n\pi}\right),\quad y=a\left(1-\frac{\theta\cos\theta}{2n\pi}\right),\ \text{for required length.}$$

$$s=a\int_0^{2\pi/m}\sqrt{1+\frac{1+\theta^2}{4\pi^2n^2}-\frac{\sin\theta+\theta\cos\theta}{\pi n}}\,d\theta. \qquad \text{F.}$$

192. Proposed by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Show that the volume V of the hyper-ellipsoid with semi-axes a_1, a_2, a_3, a_4, etc., in space of $2n$ and $2n+1$ dimensions is

$$V_{2n}=\frac{a_1.a_2.a_3\ldots\ldots a_{2n}.\pi^n}{1.2.3.4.5\ldots\ldots n};\qquad V_{2n+1}=\frac{2^{n+1}.a_1.a_2.a_3\ldots\ldots a_{2n+1}.\pi^n}{1.3.5.7.9\ldots\ldots(2n+1)}.$$

Solution by the PROPOSER.

Let $\left(\frac{x_1}{a_1}\right)^2+\left(\frac{x_2}{a_2}\right)^2+\left(\frac{x_3}{a_3}\right)^2+\ldots\ldots+\left(\frac{x_r}{a_r}\right)^2=1$ be the equation to the hyper-ellipsoid. Then its volume is $V=2^r\int\int\int\ldots\ldots dx_1\,dx_2\,dx_3\ldots\ldots dx_r$.

Let $x_1/a_1=y_1$, $x_2/a_2=y_2$, ..., $x_r/a_r=y_r$.

$\therefore V=2^ra_1a_2a_3\ldots\ldots a_r\int\int\int\ldots\ldots dy_1\,dy_2\,dy_3\ldots\ldots dy_r$, subject to the condition, $y_1^2+y_2^2+y_3^2+\ldots\ldots+y_r^2=1$.

$\therefore V=\dfrac{a_1a_2a_3\ldots\ldots a_n[\Gamma(\frac{1}{2})]^r}{\Gamma(1+\frac{1}{2}r)}$.

When $r=2n$,

$$V=\frac{a_1a_2a_3\ldots\ldots a_{2n}\,\pi^n}{1.2.3.4\ldots\ldots n}.$$

When $r=2n+1$,

$$V=\frac{a_1a_2a_3\ldots\ldots a_{2n+1}\,\pi^n\,\Gamma(\frac{1}{2})}{\frac{1}{2}.\frac{3}{2}.\frac{5}{2}\ldots\ldots\frac{2n+1}{2}\Gamma(\frac{1}{2})}=\frac{2^{n+1}a_1a_2a_3\ldots\ldots a_{2n+1}\,\pi^n}{1.3.5.7.9\ldots\ldots(2n+1)}.$$

193. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

Find the eccentricity of the maximum semi-ellipse inscribed in a given isosceles triangle.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va., and J. SCHEFFER, Hagerstown, Md.

Let the mid-point of the base be the origin, a=altitude, b=base of triangle. Let $x^2/m^2+y^2/n^2=1$ be the ellipse. Then πmn=maximum.

$\therefore n/m=y/x$.

Let (h, k) be the tangent point of the ellipse with a side.

Then $a=n^2/k$, $\frac{1}{2}b=m^2/n$, or $k/h=2a/b=n/m$. Also $2ah+bk=hk$.

$\therefore k=\frac{1}{2}a$, $h=\frac{1}{4}b$. $\therefore b^2n^2+4a^2m^2=16m^2n^2$. And $bd=2am$.

$$\therefore m=\frac{b}{2\sqrt{2}},\ n=\frac{a}{\sqrt{2}};\ (\text{eccentricity})^2=\frac{4a^2-b^2}{4a^2}.$$

II. Solution by A. H. HOLMES, Brunswick, Me.

Let $2a$=base of the isosceles triangle, and b its perpendicular height.

Construct on $2a$ an equilateral triangle, and inscribe in it a semi-circle its diameter collinear with base $2a$. Then the radius of the semi-circle will be $\frac{a\sqrt{3}}{2}$ which is one-half the perpendicular of the equilateral triangle. Now consider this triangle to be projected into an isosceles triangle whose base will be, of course, the same as that of the equilateral triangle, but whose perpendicular height is b. The semi-circle inscribed in the equilateral triangle will be projected into the maximum semi-ellipse that can be inscribed in the isosceles triangle, and one of its semi-axes will have the same proportion to the perpendicular of the isosceles triangle that the radius of the semi-circle has to the perpendicular of the equilateral triangle.

$\therefore$ Eccentricity of ellipse $=\dfrac{\sqrt{(b^2-3a^2)}}{b}$ or $\dfrac{\sqrt{(3a^2-b^2)}}{a\sqrt{3}}$, accordingly as $\sqrt{(a^2+b^2)}$ is greater or less than $2a$. If b=one of the sides,

$$\varepsilon=\sqrt{\frac{b^2-4a}{b^2-a^2}},\ \text{or}\ \frac{\sqrt{(4a^2-b^2)}}{a\sqrt{3}}.$$

Also solved by Jacob Westlund.

DIOPHANTINE ANALYSIS.

123. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

Of two numbers $a_ib_ic_id_ie_i$ $(i=1, 2)$ it is given that their 10 digits $a_1, \ldots, e_2$ form a permutation of 0, 1, ..., 9, and that the sum of the two is $x3951$. Give an immediate evaluation of x; also list the possible pairs $a_1, a_2; \ldots; e_1, e_2$.

Solution by the PROPOSER.

Since the sum of the 10 digits is 45, $x+18$ must be a multiple of 9 by the rule of casting out of 9's. Hence $x=9$.

Next, on adding the third column there cannot be 1 to carry; otherwise c_1+c_2 or c_1+c_2+1 would be 19, and $c_1 \leqq 9$, $c_2 \leqq 9$. Hence

$$(1)\quad b_1+b_2=3,\ a_1+a_2=9;\ \text{or}\ (2)\quad b_1+b_2=13,\ a_1+a_2=8.$$

If $e_1+e_2=1$, the b's are not 0, 3; nor 1, 2. Hence in this case,

$e_1, e_2=0, 1$; $b_1+b_2=13$; $a_1+a_2=8$; $d_1+d_2=5$, $c_1+c_2=9$; or $d_1+d_2=15$, $c_1+c_2=8$.

Thus $a_1, a_2=2, 6$ or $3, 5$; $d_1+d_2=15$, giving sets I, II below. Let next $e_1+e_2=11$. Then $d_1+d_2=4$, $c_1+c_2=9$; or $d_1+d_2=14$, $c_1+c_2=8$. From these and (1) or (2), we get III.....XI as the only sets.

	a_1, a_2	b_1, b_2	c_1, c_2	d_1, d_2	e_1, e_2
I	2, 6	4, 9	3, 5	7, 8	0, 1
II	3, 5	4, 9	2, 6	7, 8	0, 1
III	1, 8	0, 3	2, 6	5, 9	4, 7
IV	3, 6	1, 2	0, 8	5, 9	4, 7
V	4, 5	0, 3	1, 7	6, 8	2, 9
VI	0, 9	1, 2	3, 5	6, 8	4, 7
VII	1, 7	5, 8	3, 6	0, 4	2, 9
VIII	2, 6	5, 8	0, 9	1, 3	4, 7
IX	3, 5	6, 7	1, 8	0, 4	2, 9
X	0, 8	4, 9	2, 7	1, 3	5, 6
XI	0, 8	6, 7	4, 5	1, 3	2, 9

Sets VI, X, and XI may properly be excluded.

Also solved by G. B. M. Zerr.

MISCELLANEOUS.

132. Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.

Six officers of different grades (1, 2, 3, 4, 5, 6) from each of six branches of the army (a, b, c, d, e, f) are to be arranged in a square so that each rank and each file shall have an officer of each grade and each branch. Can it be done? If not, prove it. The arrangement of five officers of each kind is easy.

Remark by L. E. DICKSON, The University of Chicago.

This problem, proposed in the February, 1903, number, is here repeated to call attention to the fact that no solution has yet been sent to the editors. If, instead of 6, we employ an odd number n, we obtain an immediate solution with $a_1b_2c_3.....\nu_n$ as the first row, $a_1a_na_{n-1}.....a_3a_2$ as the main diagonal, the scheme being completed by permuting $a, b, c, ..., \nu$ cyclically, and $1, 2, ..., n$ cyclically. Thus for $n=3$ we obtain the (single, notation apart) possible solution:

$$\begin{array}{ccc} a_1 & b_2 & c_3 \\ c_2 & a_3 & b_1 \\ b_3 & c_1 & a_2. \end{array}$$

The problem is impossible for $n=2$. I proceed to show that there are exactly two distinct solutions for $n=4$. I first find the possible schemes for the *letters*.

By interchange of columns, we may bring the a's into the main diagonal. Call the first row $abcd$. If b is fourth in the second row, we interchange the third and fourth row, the third and fourth columns, and permute c, d, and get $abcd$ as the new first row, a's in diagonal, and b third in second row. The scheme is then necessarily (I). If b is first in the second row, the scheme is either (II) or (III).

$$\text{(I)}\ \begin{matrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{matrix} \qquad \text{(II)}\ \begin{matrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{matrix} \qquad \text{(III)}\ \begin{matrix} a & b & c & d \\ b & a & d & c \\ d & c & a & b \\ c & d & b & a \end{matrix}$$

If in (III) we interchange the second and third columns, and also rows, and permute b, c, we get (I).

We attach the subscripts to the letters of the first row in the order 1, 2, 3, 4. The diagonal terms must be $a_1a_4a_2a_3$ or $a_1a_3a_4a_2$. For (I), b in the second row must be b_1; in the former case, d in the fourth row must be d_4 contrary to d of the first row; in the latter case, d in the fourth row must be d_2, contrary to a of the fourth row. Hence (I) is excluded. For (II) the two schemes are evidently

$$(A)\ \begin{matrix} a_1 & b_2 & c_3 & d_4 \\ b_3 & a_4 & d_1 & c_2 \\ c_4 & d_3 & a_2 & b_1 \\ d_2 & c_1 & b_4 & a_3 \end{matrix} \qquad (B)\ \begin{matrix} a_1 & b_2 & c_3 & d_4 \\ b_4 & a_3 & d_2 & c_1 \\ c_2 & d_1 & a_4 & b_3 \\ d_3 & c_4 & b_1 & a_2 \end{matrix}$$

If we view the square (A) from the side, instead of the top, we get (B). If we reflect (A) on the main diagonal and then permute 2, 4, 3 cyclically, we obtain (B). But by no change of notation of letters or subscripts is (A) converted into (B). Note that the arrangements of the letters (as well as the subscripts) in (A) or (B) define the non-cyclic group of order 4.

These results for $n=4$ and n odd suggest that for $n=6$ the arrangements of the letters and subscripts might be derivable from those in the first row by means of the substitutions of the same regular group on six letters. But this is readily verified to be impossible. Hence if there is a solution for $n=6$, it is not a group solution of the type mentioned.

PROBLEMS FOR SOLUTION.

ALGEBRA.

228. **Proposed by G. W. GREENWOOD, M. A. (Oxon), Professor of Mathematics, McKendree College, Lebanon, Ill.**

Sum the infinite series

$$\frac{1}{11.13}+\frac{1}{23.25}+\frac{1}{35.37}+\frac{1}{47.49}+\frac{1}{59.61}+\text{.........} \quad \text{[Oxford, 1895].}$$

229. Proposed by B. F. YANNEY, Mount Union College, Alliance, O.

If $a_1{}^n+a_2{}^n+a_3{}^n+\ldots\ldots+a_r{}^n=A^n$, $a_1{}^m+a_2{}^m+a_3{}^m+\ldots\ldots+a_r{}^m>$ or $<A^m$, according as $m<$ or $>n$; provided all the letters stand for positive real numbers.

230. Proposed by G. W. GREENWOOD, M. A. (Oxon), Lebanon, Ill.

Find the value of the determinant of n rows,

$$\begin{vmatrix} 5 & 2 & 0 & 0 & 0 & \ldots\ldots \\ 2 & 5 & 2 & 0 & 0 & \ldots\ldots \\ 0 & 2 & 5 & 2 & 0 & \ldots\ldots \\ 0 & 0 & 2 & 5 & 2 & \ldots\ldots \\ 0 & 0 & 0 & 2 & 5 & \ldots\ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\ldots \end{vmatrix}$$

[Oxford, 1900.]

231. Proposed by O. L. CALLECOT, Omaha, Neb.

Sum to infinity: $\frac{1}{2.3.4}+\frac{1}{5.6.7}+\frac{1}{8.9.10}+\ldots\ldots$

232. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

If one person out of 50 die annually and one person out of 30 is born annually, how long at this rate would be required for the population to treble itself?

233. Proposed by J. J. KEYES, Fogg High School, Nashville, Tenn.

At what time between 10 and 11 o'clock is the second hand of a clock one minute space nearer to the hour hand than it is to the minute hand?

GEOMETRY.

254. Proposed by W. J. GREENSTREET, M. A., Editor of the Mathematical Gazette, Stroud, England.

Find the cartesian equation to a line that is both tangent and normal to the cardioid.

255. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Find the envelope of the straight line that connects the extremities of two conjugate diameters of an ellipse.

256. Proposed by F. P. MATZ, Ph. D., Sc. D., Reading, Pa.

The bisectors of the four angles of any quadrilateral intersect in four points, all of which lie on the circumference of the same circle.

CALCULUS.

194. Proposed by G. W. GREENWOOD, M. A. (Oxon), Lebanon, Ill.

Show that the volume of the solid generated by the revolution of a segment of a circle, less than a semi-circle, about the diameter parallel to its chord, is equal to that of a sphere having a diameter equal to the chord; and hence that the volume is independent of the magnitude of the original circle, the length of the chord being known.

195. Proposed by CHRISTIAN HORNUNG, Heidelberg University, Tiffin, O.

Given a right cone of altitude h and radius r, to locate the plane parallel to its side which bisects the cone.

MECHANICS.

175. Proposed by J. F. LAWRENCE, A. B., Professor of Mathematics, Oklahoma Agricultural College, Stillwater, Oklahoma.

A cylinder descends down a plane, the inclination of which to the horizon is α, unwrapping a fine string fixed at the highest point of the plane. Find the angle through which the plane must be depressed in order that a sphere, descending under like circumstances, may experience the same acceleration.

176. Proposed by A. H. HOLMES, Brunswick, Me.

A solid cube weighs 300 pounds. If a power is applied at an angle of 45° at an upper edge of the cube, how many foot-pounds will be required to overturn the cube?

DIOPHANTINE ANALYSIS.

126. Proposed by R. A. THOMPSON, M. A., C. E., Engineer Railroad Commission of Texas.

Eight persons wish to play a series of games of progressive duplicate whist. In one evening, 12 boards are played, 4 boards (and return) by one couple against each of the other three couples, the same partners being retained throughout one evening. How many evenings will be required to complete the series, and what is the order of play, it being required that each player shall play with every other player as partner, and that each couple shall play once and but once against every other couple.

AVERAGE AND PROBABILITY.

162. Proposed by F. P. MATZ, Ph. D., Sc. D., Reading, Pa.

Two points are taken at random in the surface of a circle and a chord is drawn through them. Find the average area of the segment containing the center of the circle.

GROUP THEORY.

7. Proposed by M. E. GRABER, A. M., Heidelberg University, Tiffin, Ohio.

Which linear substitution will transform $x_1x_2+x_3x_4+x_5x_6=0$ into $y_1^2+y_2^2+y_3^2-y_4^2-y_5y_6=0$?

MISCELLANEOUS.

147. Proposed by F. P. MATZ, Ph. D., Sc. D., Reading, Pa.

If P be a point within the scalene triangle, such that $\angle PAB=\angle PBC=\angle PCA=\psi$, then $\cot\psi=\cot A+\cot B+\cot C$.........(1), and $\operatorname{cosec}^2\psi=\operatorname{cosec}^2 A+\operatorname{cosec}^2 B+\operatorname{cosec}^2 C$.........(2).

NOTE.—Problems and solutions in the departments of Geometry, Calculus, Mechanics, and Average and Probability should be sent to B. F. Finkel; and those in the departments of Algebra, Diophantine Analysis, Miscellaneous, and Group Theory should be sent to Dr. Saul Epsteen. Our contributors should carefully observe this notice if proper credit for contributions is to be given.

NOTES.

A list of one hundred mathematical models, made and for sale by Mr. R. P. Baker, 5519 Monroe Street, Chicago, Ill., has recently been issued. The models relate to solid geometry, linkages, crystallography, twisted cubics, cubic cones, scrolls, surfaces of the second order, etc. In view of the numerous orders received, Mr. Baker expects to devote his entire attention to the construction of models. D.

F. Strobel of Jena, has compiled a directory of all living mathematicians, physicists, astronomers, and chemists. It will be published by the firm of J. A. Barth of Leipzig, and revised every two years. S.

Mr. J. R. Hogan and Mr. E. Whitford have been appointed tutors in mathematics at the College of the City of New York. S.

The medal of the Royal Society of London was awarded to Professor W. Burnside for his researches on the theory of groups. S.

BOOKS.

A College Algebra. Seventh Edition. By J. M. Taylor, A. M., LL. D., Professor of Mathematics in Colgate University. Boston and Chicago: Allyn and Bacon. 363 pages.

To the introductory work, covering the ground of a high school course, the author devotes the first hundred pages, the remainder of the book being devoted to subjects adapted to the first year at college. In Chapter XII the fundamental notion of *functionality* is introduced and briefly illustrated by means of simple examples. In this chapter the theory of limits is also developed.

One of the chief merits of the book consists, in the opinion of the reviewer, of the introduction of the chapter on the derivatives of algebraic functions. The chapter on the development of functions in series, on convergency and divergency, logarithms and theory of equations, are written in Dr. Taylor's inimitable style. Chapter XVII on compound interest and annuities, however, treats the latter subject in the brief manner of most algebras, the annuities there considered are *annuities certain* and not *contingent annuities based upon a mortality table.* S. E.

The Essentials of Algebra. For Secondary Schools. By Robert J. Aley, Ph. D., and David R. Rothrock, Ph. D., Professors in the University of Indiana. Silver, Burdett & Co. 1904. 295 + vii pages.

It was to be expected that as soon as the laboratory method of teaching mathematics had been sufficiently developed, text-books adapted to this form of instruction would make their appearance. The present book is the first of this kind, and is exceedingly well adapted to laboratory courses in secondary schools. As might be expected under the circumstances, the striking feature is the concreteness with which the subject is treated, principally through the chapters on graphic methods. Some of the most commendable characteristics of the book are the frequency with which diagrams are introduced, the explanation of Pascal's Triangle in connection with the binomial theorem, and Argand's representation of $i=\surd(-1)$.

The value of the book is enhanced and the pages rendered attractive to the eye by an excellent index, illustrative solutions of problems, and the frequent use of three different kinds of type.

In the opinion of the reviewer, the words "variable" and "constant" (p. 15, *et seq*) in the sense used are unfortunate; the words "unknown" and "parameter" being more suitable for the purpose. As the text explains (p. 205) $i=\surd(-1)$ may be interpreted as the unit on the axis at right angles to the axis of reals. Therefore, the term "imaginary" while sanctioned by usage and history, is undesirable. i is best regarded as the special complex number $a+bi$, where $a=0$, $b=1$.

The authors enunciate without proof the theorem that the graph of a linear equation in two variables is a straight line,—probably with the idea that this proof should be delayed to a later period in the course, and that the young student feels intuitively convinced of their truth after having constructed the graphs of several such equations.

We are indebted to the authors for an excellent text-book which combines the merits of the older texts with the recent advances in the pedagogy of mathematics.

CHICAGO, *March, 1905.* ALMA E. KLUNDER.

Elements of Mechanics. Forty Lessons for Beginners in Engineering. By Mansfield Merriman, Professor of Civil Engineering in Lehigh University. 12mo, 172 pages, 142 figures. Cloth, $1.00 net. New York: John Wiley & Sons.

The aim of this volume is the application of the best methods of applied mechanics to the development of the fundamental principles and methods of rational mechanics.

"To this end, constant appeals are made to experience, by which alone the laws of mechanics can be established, numerous numerical problems are stated as exercises for the student, and a system of units is employed with which every boy is acquainted." *Preface.* The book is one that will be useful in establishing the fundamental principles of theoretical and practical mechanics. B. F. F.

5

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NOTE ON GROUPS OF ORDER* p^2q^2.

By O. E. GLENN, The University of Pennsylvania.

The purpose of this note is to prove the following theorem:

If a group G of order p^2q^2 $(p>q)$ has five distinct series of composition, all arrangements of the composition factors excepting (q, p, p, q) being posssble, then must $q=2$ and $p=3$. The only existent group is thus of order 36, *and it is the direct product of the tetrahedron-group and a cyclic group of order* 3.

The invariant subgroup Hq^2p is Abelian, and Hp^2q is a divisible type $\{S_1, S_3\}\{S_2\}$ defined by

$$S_1{}^p=S_2{}^p=S_3{}^q=1, \quad S_1S_2=S_2S_1, \quad S_3S_2=S_2S_3, \quad S_3{}'S_1S_3=S_1{}^a.$$

The Sylow subgroup $I_{q^2}=\{S_3, S_4\}$ is invariant under H_{q^2p} and hence under G. Likewise $\{S_1, S_2\}$ is self-conjugate in G, and these two subgroups contain all the subgroups of G of orders q and p, respectively. The operator S_1 transforms $\{S_3\}$ into p conjugates within $\{S_3, S_4\}$, and since the number of these cannot exceed

$$N_q=\frac{q^2-1}{q-1}=q+1$$

it follows that $p=q+1$, and therefore $p=3$, $q=2$.

Hence $a\equiv-1 \pmod 3$, and the $N_2=3$ subgroups are

$$\{S_3\}, \quad \{S_4\}, \text{ and } \{S_3S_4\}.$$

*Le Vavavasseur has given in Comptes Rendus, Vol. 128 (1899), p. 152, a list of the groups of order p^2q^2, the proofs having been suppressed. [ED. D.]